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THE MODEL THEORY OF DIFFERENTIAL FIELDS REVISITED†

BY

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ABSTRACT

The intent of this article is to provide a general and elementary account of the model theory of differential fields, collecting together various results (many without proof) and offering a few algebraic details for the logician reader. The first model-theoretic look at differential fields was taken by Abraham Robinson in the context of model completeness, while later developments have served to illustrate concepts developed by Morley and Shelah.

1. Algebraic preliminaries

We begin with a somewhat leisurely account of algebraic facts, with an eye for those phenomena of model-theoretic importance.

DEFINITION. An (ordinary) differential field $\mathcal{F}$ is a field in the usual algebraic sense together with a derivation D on $\mathcal{F}$, i.e., D is a unary function satisfying $D(a+b) = Da + Db$ and $D(ab) = aDb + (Da)b$ for all $a, b \in \mathcal{F}$.

Although we restrict our attention to ordinary differential fields, it is not difficult to see that the results generalize to partial differential fields (ones with a finite set of commuting derivations). In distinguishing between differential fields and fields we shall frequently refer to the latter as algebraic fields.

A basic motivating example of a differential field is the field of all complex meromorphic functions in one complex variable x on some fixed region, with D taken as d/dx.

For any differential field $\mathcal{F}$ the set $\mathcal{C}$ of constants ($\mathcal{C} = \{a \in \mathcal{F} \mid Da = 0\}$) is a sub-differential field of $\mathcal{F}$; $\mathcal{C}$ contains the prime (algebraic) field of $\mathcal{F}$ as well as anything separably algebraic over the prime field, as is readily checked. In characteristic $p \neq 0$ we have $\mathcal{F}^p \subseteq \mathcal{C}$, since $D(a^p) = pa^{p-1}Da = 0$ for all $a \in \mathcal{F}$.

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A differential field is called *differentially perfect* if either $p = 0$ or $\mathcal{F}^p = \mathcal{C}$, where p is the characteristic of $\mathcal{F}$. Note that if $\mathcal{F}$ is its own constant field, then $\mathcal{F}$ is differentially perfect just in case $\mathcal{F}$ is algebraically perfect. Furthermore any differential field can be extended (but not, in general, in any unique fashion) to a differentially perfect field by successive adjunctions of pth roots for all constants.

Some basic references for differential algebra are Ritt [7], Kaplansky [3] and Kolchin [5]; everything in this section can be found in Kolchin's book.

In considering both model completeness and stability results it will be useful to have the notion of differentially algebraic and the associated notions of differential polynomials and ideals over a differential field $\mathcal{F}$. We deal for the most part with simple differential extensions, both for ease of notation and because certain model theoretic considerations make it sufficient to look only at simple extensions in the characteristic 0 case. For nonzero characteristic the additional complications have been described elsewhere [24] and we give them only cursory treatment here.

DEFINITION. Let a be an element of some differential field extension $\mathcal{F}'$ of $\mathcal{F}$. We say a is *differentially algebraic* over $\mathcal{F}$ if the set $\{a, Da, D^2a, \cdots\}$ is algebraically dependent over $\mathcal{F}$. Otherwise a is *differentially transcendental* over $\mathcal{F}$. We denote by $\mathcal{F}\langle a\rangle$ the differential field extension of $\mathcal{F}$ generated by a; $\mathcal{F}\langle a\rangle = \mathcal{F}(a, Da, \cdots)$ as algebraic field, with D given by restriction from $\mathcal{F}'$.

The *differential polynomial ring* $\mathcal{F}\{y\}$ *in one differential indeterminate* y is the algebraic ring $\mathcal{F}[y, Dy, D^2y, \cdots]$ in algebraic indeterminates $y = D^0y, D^1y, D^2y, \cdots$ over $\mathcal{F}$, with differential ring structure given by extending D on $\mathcal{F}$ such that $D(D^ny) = D^{n+1}y$ for all $n = 0, 1, \cdots$. A *differential ideal* $\mathcal{I}$ in $\mathcal{F}\{y\}$ is an algebraic ideal which is closed under D.

Given $f \in \mathcal{F}\{y\}$, $f \neq 0$, the *order* of f is the greatest n, if any, such that D^ny occurs nontrivially in f; if $f \in \mathcal{F}$, $f \neq 0$, the *order* of f is -1. Given f of order $n \geqq 0$ we write $f = \Sigma_{i=0}^m f_i(D^ny)^i$ where each $f_i \in \mathcal{F}[y, \cdots, D^{n-1}y]$, $m > 0$ and $f_m \neq 0$. The *separant* S_f of f is the partial derivative of f with respect to D^ny, i.e., $S_f = \Sigma_{i=1}^m if_i(D^ny)^{i-1}$, and the *initial* I_f of f is the leading coefficient f_m of f as a polynomial in D^ny. Observe that $I_f \neq 0$ and, in characteristic 0, $S_f \neq 0$ for any f of order $\geqq 0$.

DEFINITION. A partial ordering on nonzero elements of $\mathcal{F}\{y\}$ is defined as follows:

f is lower than g if the order of f is less than the order of g, or if $0 \leqq n =$ order $f =$ order g and f is of lower degree than g as a polynomial in D^ny. Observe that S_f, if nonzero, and I_f are each lower than f.

Let $[f]$ denote the differential ideal generated by $f \in \mathcal{F}\{y\}$, so that $[f] = (f, Df, \cdots)$ as algebraic ideal. Then the usual division algorithm procedure for polynomials in several variables (p. 6 of Ritt [7]) yields that for any $g \in \mathcal{F}\{y\}$ there is g_0 lower than f or zero and integers $j, k \geqq 0$ such that $S_f^j I_f^k g = g_0 \in [f]$. We call such a g_0 a *remainder of g with respect to f.*

Now suppose $f = f(y, Dy, \cdots, D^n y) \in \mathcal{F}\{y\}$, f of order $n \geqq 0$, f irreducible (as a polynomial in $\mathcal{F}[y, \cdots, D^n y]$) such that $S_f \neq 0$. Let $a = a_0, a_1, \cdots, a_{n-1}$ be algebraic indeterminates over $\mathcal{F}$ and let a_n be a root of $f(a_0, \cdots, a_{n-1}, X)$ over the algebraic field $\mathcal{F}(a_0, \cdots, a_{n-1})$, where we note that $f(a_0, \cdots, a_{n-1}, X)$ is irreducible and separable (since $S_f \neq 0$) as an algebraic polynomial in X over $\mathcal{F}(a_0, \cdots, a_{n-1})$. A differential field extension $\mathcal{F}\langle a \rangle$ of $\mathcal{F}$ is obtained by taking $\mathcal{F}\langle a \rangle = \mathcal{F}(a_0, \cdots, a_{n-1}, a_n)$ as fields, and extending D from $\mathcal{F}$ to $\mathcal{F}\langle a \rangle$ by assigning $Da_i = a_{i+1}$, $i = 0, \cdots, n-1$. This uniquely defines a differential field (see, for example, Seidenberg [16, p. 179] or Weil [22]). Observe that $f(a) = 0$ while $g(a) \neq 0$ for all $g(y) \in \mathcal{F}\{y\}$ such that g is lower than f. The element a is called a *generic zero* of f over $\mathcal{F}$.

If we take $\mathcal{I} = \{g \in \mathcal{F}\{y\} \mid g(a) = 0\}$ then $\mathcal{I}$ is a prime differential ideal (prime in the usual sense) but it is not however the case that $\mathcal{I}$ is the differential ideal (or even the radical of the differential ideal) generated by f. Here $\mathcal{I}$ is called the *defining differential ideal* of a over $\mathcal{F}$.

Conversely, given any extension $\mathcal{F}\langle a \rangle$ of $\mathcal{F}$ where a is differentially algebraic over $\mathcal{F}$, there is a lowest irreducible f, unique up to a nonzero factor in $\mathcal{F}$, such that $f(a) = 0$. If $S_f \neq 0$ (which always is the case for irreducible f associated with differentially algebraic a if $\mathcal{F}$ is differentially perfect) then we again know that $\mathcal{F}\langle a \rangle$ is determined up to differential field isomorphism over $\mathcal{F}$ by the fact that a is a generic zero of f, i.e., by the equation $f(a) = 0$ together with $g(a) \neq 0$ for all lower g (in fact, it suffices to know this for g of lower order than that of f).

Finally, if $\mathcal{F}\langle a \rangle$ and $\mathcal{F}\langle b \rangle$ are differential field extensions of $\mathcal{F}$ with a and b each differentially transcendental over $\mathcal{F}$, then, for the usual algebraic reasons, $\mathcal{F}\langle a \rangle$ and $\mathcal{F}\langle b \rangle$ are isomorphic over $\mathcal{F}$ under a differential field isomorphism sending a to b. Thus we have a reasonable idea of all simple extensions over differentially perfect fields. We note that the constant fields of such extensions are not easily identifiable in general; one exception is that of a differentially transcendental extension $\mathcal{F}\langle a \rangle$ of $\mathcal{F}$. There the constant field of $\mathcal{F}\langle a \rangle$ is $\mathcal{C}(a^p, (Da)^p, \cdots)$ where $\mathcal{C}$ is the constant field of $\mathcal{F}$ and where p is the characteristic of $\mathcal{F}$ (so in characteristic $p = 0$ we have simply $\mathcal{C}$ as constant field). Thus we have the following.

THEOREM 1.1. *Over a differentially perfect field $\mathcal{F}$, there are* $\max\{\aleph_0, \operatorname{card}\mathcal{F}\}$ *isomorphism types of elements over $\mathcal{F}$.*

PROOF. Any simple extension $\mathcal{F}\langle a\rangle$ of a differentially perfect field $\mathcal{F}$ is either differentially transcendental or is determined by the fact that a is a generic zero of an irreducible differential polynomial $f \in \mathcal{F}\{y\}$. There are $\max\{\aleph_0, \operatorname{card}\mathcal{F}\}$ such f's and only one differentially transcendental extension, so there are at most $\max\{\aleph_0, \operatorname{card}\mathcal{F}\}$ such extensions. It is clear also that there are $\max\{\aleph_0, \operatorname{card}\mathcal{F}\}$ distinct isomorphism types of elements: if $\mathcal{F}$ is finite there are enough types in its algebraic closure, if $\mathcal{F}$ is infinite, there are $\operatorname{card}\mathcal{F}$ many monic irreducible polynomials of the form $Dy - b$ for $b \in \mathcal{F}$, each of which gives a distinct isomorphism type over $\mathcal{F}$.

To describe the isomorphism type of a simple differentially transcendental extension it is clear that no finite number of (differential polynomial) equations and inequations suffices, nor even (in contrast now to a simple differentially algebraic extension) an arbitrary number of equations and inequations in finitely many variables $y, Dy, \cdots, D^n y$; thus the type of a transcendental extension cannot be principal (in the model theoretic sense). To identify which of the differentially algebraic types are principal we use the following notion. Throughout we assume $\mathcal{F}$ to be differentially perfect.

DEFINITION. Let $\mathcal{I}$ be a differential ideal in $\mathcal{F}\{y\}$. $\mathcal{I}$ is *perfect* if $f^n \in \mathcal{I}$ implies $f \in \mathcal{I}$, for all $f \in \mathcal{F}\{y\}$ and $n > 0$. $\mathcal{I}$ is *constrained* if $\mathcal{I}$ is perfect and there exists $g \in \mathcal{F}\{y\} - \mathcal{I}$ such that $\mathcal{I}$ is maximal among perfect differential ideals with respect to exclusion of g; i.e., if $\mathcal{I} \subsetneqq \mathcal{I}'$, $\mathcal{I}'$ a perfect differential ideal in $\mathcal{F}\{y\}$, then $g \in \mathcal{I}'$. Such a g is called a *constraint* of $\mathcal{I}$. A *constrained element a* in an extension of $\mathcal{F}$ is an element whose defining differential ideal is constrained.

REMARKS. 1. A constrained ideal is always prime.

2. Given any ideal $\mathcal{I}_0$ which is the defining differential ideal of some element over $\mathcal{F}$, and any $g \notin \mathcal{I}_0$, there exists $\mathcal{I} \supseteq \mathcal{I}_0$ such that $\mathcal{I}$ is constrained with constraint g.

3. If a is constrained over $\mathcal{F}$, $\mathcal{F}$ differentially perfect, then $\mathcal{F}\langle a\rangle$ is differentially perfect (see Wood [24]).

THEOREM 1.2. *Let $\mathcal{I}$ be constrained with constraint g over a differentially perfect differential field $\mathcal{F}$, and let f be a lowest element of $\mathcal{I}$. Then for $a \in \mathcal{F}' \supseteq \mathcal{F}$, $\mathcal{F}'$ a differential field extension of $\mathcal{F}$, the following are equivalent:*

1) *$\mathcal{I} = \{h \in \mathcal{F}\{y\} \mid h(a) = 0\}$, i.e., $\mathcal{I}$ is the defining differential ideal of a.*

2) *a is a generic zero of f.*

3) *$f(a)=0$ and $(S_fI_fg)(a)\neq 0$.*

PROOF. By Remark 1 above, f is irreducible.

(1) $\Rightarrow$ (2). Clearly if $\mathcal{I}$ is the defining differential ideal of a, then $f(a)=0$ since f was chosen to be in $\mathcal{I}$. If h is lower than f then $h\notin\mathcal{I}$ hence $h(a)\neq 0$. Thus a is a generic zero of f.

(2) $\Rightarrow$ (3). If a is a generic zero of f then clearly $f(a)=0$. Now we can take g_0 a remainder of g with respect to f, so that $S_f^jI_f^kg-g_0\in[f]$ where g_0 is lower than f. If $g_0=0$ then $S_f^jI_f^kg\in\mathcal{I}$. Since $\mathcal{I}$ is prime this implies $S_f\in\mathcal{I}$ or $I_f\in\mathcal{I}$ or $g\in\mathcal{I}$, each of which is impossible since S_f and I_f are lower than f, $S_fI_f\neq 0$, and $g\notin\mathcal{I}$. Thus $g_0\neq 0$ and so $(S_f^jI_f^kg)(a)\neq 0$.

(3) $\Rightarrow$ (1). Let $f(a)=0$ and $(S_fI_fg)(a)\neq 0$. Let $h\in\mathcal{F}\{y\}$ and find h_0 such that $S_f^jI_f^kh-h_0\in[f]$ where h_0 is lower than f. If $h\in\mathcal{I}$ then $h_0\in\mathcal{I}$ and so $h_0=0$ by our choice of f. Thus $(S_fI_fh)(a)=0$. But $(S_fI_f)(a)\neq 0$ by assumption, so $h(a)=0$. Thus the defining differential ideal $\mathcal{I}'$ of a contains $\mathcal{I}$ and does not contain g (since $g(a)\neq 0$). By maximality of $\mathcal{I}$ such that $g\notin\mathcal{I}$ we have $\mathcal{I}=\mathcal{I}'$ and (1) is proved.

By the above result we see that for an element a which is constrained over the differentially perfect differential field $\mathcal{F}$, the isomorphism type of a is given by one equation $f(a)=0$ and one inequation $(S_fI_fg)(a)\neq 0$. By the division algorithm procedure we may replace S_fI_fg by a differential polynomial lower than f, and then by the division algorithm for f and g as polynomials in D^ny over $\mathcal{F}[y,\cdots,D^{n-1}y]$ (where n is the order of f) we can find g_0 of order less than n such that $f(y)=0$, $g_0(y)\neq 0$ is equivalent to $f(y)=0$, $(S_fI_fg)(y)\neq 0$. Thus the isomorphism type of a constrained element is determined by a pair $f(y)=0$, $g(y)\neq 0$ where g has lower order than f. This is not to say that every such pair determines an isomorphism type, but it is correct that every such pair ($f(y)=0$, $g(y)\neq 0$ with S_f and g nonzero, and order $f>$ order g) has a solution in some extension of $\mathcal{F}$: a generic zero of any irreducible factor f_0 of f with the same order as f will do. In fact we can solve the pair by a constrained element by taking $\mathcal{I}$ to be any ideal containing f and constrained with constraint g, and then letting a be an element as in Theorem 1.2. It is also an immediate consequence that given $f(y)=0$, $g(y)\neq 0$ with $f,g\in\mathcal{F}\{y\}$, $S_fg\neq 0$, and order $f>$ order g, this system always has a solution not only over $\mathcal{F}$ but over any prescribed extension $\mathcal{F}'\supseteq\mathcal{F}$. We state without proof the following theorem of a primitive element, which is necessary for us only in the nonzero characteristic case. (See Seidenberg [16].)

THEOREM 1.3. *Let $\mathcal{F}$ be differentially perfect such that $\mathcal{F}$ has infinite linear* *dimension over its constant field. Then for any* $\mathcal{F}' \supseteq \mathcal{F}$ *and* $b_1, \cdots, b_n$ *in* $\mathcal{F}'$ *differentially algebraic over* $\mathcal{F}$ *there is* $a \in \mathcal{F}'$ *such that* $\mathcal{F} < b_1, \cdots, b_n > = \mathcal{F}\langle a \rangle$.

2. Basic model theoretic results

Let L be a language with constants 0 and 1, binary functions $+$ and $\cdot$, unary functions $^{-1}$, $-$, D; L is the language of fields together with an additional unary function D. In L, the theory DF of differential fields is the theory of algebraic fields plus axioms for D:

$$\forall x \forall y D(x+y) = Dx + Dy \quad \text{and} \quad D(xy) = xDy + yDx.$$

Thus DF is a universal theory. For $p = 0$ or p a prime let DF_p be the theory of differential fields of characteristic p. For $p \neq 0$ we let φ_p be the sentence in L saying that if the characteristic is p then every constant has a pth root.

$$\varphi_p = \forall x \exists y ((\underbrace{1 + \cdots + 1}_{p \text{ times}}) = 0 \wedge Dx = 0 \Rightarrow y^p = x).$$

Then $DPF = DF \cup \{\varphi_p \mid p \text{ prime}\}$ is the theory of differentially perfect fields; we denote by DPF_p the theory $DPF \cup DF_p$ of differentially perfect fields of characteristic p.

For the theory DCF of differentially closed fields we add to DPF axioms (infinitely many) stating that a solution exists for every pair $f(y) = 0$, $g(y) \neq 0$, provided that $S_f g \neq 0$ and order of $f >$ order of g.

The usual chaining procedure works here to show that DCF is consistent, in fact is model consistent relative to DF; for example, one can alternate taking differentially perfect extensions and closing under solutions of appropriate pairs $f = 0$, $g \neq 0$.

If $\mathcal{F}$ is a model of DCF and a is constrained over $\mathcal{F}$ then the isomorphism type of a over $\mathcal{F}$ is given by a pair $f = 0$, $g \neq 0$, by Theorem 1.2; since such a pair is already soluble in $\mathcal{F}$ we conclude that $a \in \mathcal{F}$. Thus we see that although any differential field has proper extensions $\mathcal{F}' \supsetneq \mathcal{F}$ which contain new elements differentially algebraic over $\mathcal{F}$, none of these can be constrained over $\mathcal{F}$, whence the name constrainedly closed differential fields employed by Kolchin [4].

We note here that in particular a differentially closed field is separably algebraically closed and has separably algebraically closed constant field.

The model completeness of DCF can be proved in several ways. Robinson's original proof [9] for a theory equivalent to DCF_0 used Seidenberg's elimination theory [15], as did my proof in [23] for $p \neq 0$; there one considers differential polynomial equations and inequations in several variables and observes that Seidenberg's results imply directly that the class of existentially closed differential fields is axiomatizable. By examining the isomorphism types of simple extensions Blum [1] gave much simpler axioms for DCF_0; we shall give here a version of her proof.

Definition. A theory T has the *amalgamation property* if for any $\mathcal{A}, \mathcal{B}, \mathcal{C} \models T$ with $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A} \subseteq \mathcal{C}$ there exists a common extension $\mathcal{D}$ of $\mathcal{B}$ and $\mathcal{C}$.

For T universal, it suffices to find such a $\mathcal{D}$ for $\mathcal{B}$ and $\mathcal{C}$ of the forms $\mathcal{B} = \mathcal{A}(b)$, $\mathcal{C} = \mathcal{A}(c)$, where by $\mathcal{A}(b)$ we mean the model of T generated by $\mathcal{A} \cup \{b\}$ (for b some element of some $\mathcal{A}' \supseteq \mathcal{A}$, $\mathcal{A}' \models T$). Thus the following lemma is clear from our discussion of simple extensions.

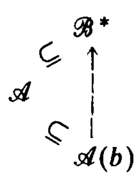
Lemma 2.1. *The theory DF_0 of differential fields of characteristic* 0 *has the amalgamation property.*

Definition. A theory T is 1-*existentially complete* if for any pair $\mathcal{A}, \mathcal{B} \neq T$ such that $\mathcal{A} \subseteq \mathcal{B}$ and for any formula φ of the form $\varphi = \exists y \psi(a_1, \cdots, a_n, y)$ where ψ is quantifier free and $a_1, \cdots, a_n \in \mathcal{A}$, if $\mathcal{B} \models \varphi$ then $\mathcal{A} \models \varphi$.

We use the following consequence of Blum's criterion for the existence of a model completion of a universal theory (theorem 17.2 of Sachs [14]) as follows:

Lemma 2.2. *Let T be a universal theory with amalgamation and let T^* be a model consistent extension of T such that T^* is* 1-*existentially closed. Then T^* is the model completion of T.*

Proof. Blum's criterion states: for T^* a model consistent extension of a universal theory T to be the model completion of T it is necessary and sufficient that any diagram of the following form be completable, for $\mathcal{A}, \mathcal{A}(b) \models T$, $\mathcal{B}^* \models T^*$, $\mathcal{B}^*$ $|\mathrm{Card}\,\mathcal{A}|^+$-saturated:



To see the lemma from this criterion, first amalgamate $\mathcal{B}^*$ and $\mathcal{A}(b)$ over $\mathcal{A}$ to get $\mathcal{C} \models T$, with

$$\begin{array}{ccccc} & \subseteq & \mathcal{B}^* & \subsetneq & \\ \mathcal{A} & & & & \mathcal{C} \\ & \subsetneq & \mathcal{A}(b) & \subseteq & \end{array}$$

Any finite set of quantifier-free formulas defined over $\mathcal{A}$ and satisfied by b in $\mathcal{A}(b)$, hence in $\mathcal{C}$, must also be satisfied in $\mathcal{B}^*$ since $\mathcal{B}^* \models T^*$ and T^* is 1-existentially closed. Since $\mathcal{B}^*$ is $|\mathrm{card}\,\mathcal{A}|^+$-saturated we know that the full set of quantifier-free formulas satisfied over $\mathcal{A}$ by b is also satisfied in $\mathcal{B}^*$ by some element b^*. The required map of $\mathcal{A}(b)$ into $\mathcal{B}^*$ is given by sending b to such an element b^*.

THEOREM 2.3. (Robinson, Blum) *The theory DCF_0 of differentially closed fields of characteristic* 0 *is the model completion of DF_0.*

PROOF. We note that DCF_0 is 1-existentially complete: this amounts to checking that any finite set of differential polynomial equations and inequations (as usual, one inequation suffices) in one variable $f_1(y) = \cdots = f_n(y) = 0$, $g(y) \neq 0$ which are satisfied by some $a \in \mathcal{F}' \supseteq \mathcal{F}$ for $\mathcal{F} \models DCF_0$, $\mathcal{F}' \models DF_0$ must already have a solution in $\mathcal{F}$. To see this, for example, we take a constrained ideal $\mathcal{I}$ over $\mathcal{F}$ with constraint g such that $\mathcal{I}$ contains the defining differential ideal of a over $\mathcal{F}$. But since $\mathcal{F} \models DCF_0$ and $\mathcal{I}$ is constrained, there is $b \in \mathcal{F}$ with ideal $\mathcal{I}$, whence b is a solution of the original system. Thus DCF_0 is 1-existentially complete. By Lemmas 2.1 and 2.2 we conclude that DCF_0 is the model completion of DF_0.

If $\mathcal{F} \models DF$, $\mathcal{F} \not\models DPF$, the characteristic p of $\mathcal{F}$ is nonzero. In this case we can find extensions $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$ with no amalgam over $\mathcal{F}$: let $c \in \mathcal{F}$ such that $D(c) = 0$, $c \notin \mathcal{F}^p$ and let $\mathcal{F}_1$ and $\mathcal{F}_2$ both have the algebraic field structure of $\mathcal{F}(c^{1/p})$, assigning $D(c^{1/p}) = 0$ in $\mathcal{F}_1$ and $D(c^{1/p}) = 1$ in $\mathcal{F}_2$. Since c has only one pth root in any extension there can be no $\mathcal{G} \models DF_p$ such that $\mathcal{F}_1 \subseteq \mathcal{G}$, $\mathcal{F}_2 \subseteq \mathcal{G}$. For a theory to have a model completion, it must itself have the amalgamation property, so the best we can get is that DPF has a model completion.

LEMMA 2.4. *For all p, DPF_p has the amalgamation property.*

PROOF. See Seidenberg [15] or Proposition 4, p. 91 of Kolchin [4]. Here we cannot consider simple extensions only, since DPF_p is not universal, and our discussion of simple extensions is thus inadequate for the present purpose.

Given 2.4 and 1.3 we can finish a proof of model completeness along the lines of 2.3. It is convenient to introduce an auxiliary language $L(r)$ where r is an unary function which will be useful in our discussion of types later, and in which we can axiomatize DPF_p with universal axioms. Let $\theta_p = \forall x((Dx \neq 0 \wedge r(x) = 0) \vee (Dx = 0 \wedge (r(x))^p = x))$. The function r is clearly definable in all models of DPF_p, so that the models of DPF_p correspond exactly to those of $DPF'_p = DF_p \cup \{\theta_p\}$, and what is now changed is that a substructure in $L(r)$ of a model of DPF'_p is again differentially perfect, while a simple extension in $L(r)$ may be much more than a simple extension in L.

THEOREM 2.5. *DCF is the model completion of DPF, hence the model companion of DF.*

PROOF. For each p we show DCF_p is the model completion of DPF_p. For $p = 0$ this is 2.3. For $p \neq 0$ we work in $L(r)$, where DPF_p is universal and has the amalgamation property by 2.4. Thus it suffices to know that DCF_p is 1-existentially complete as a theory in $L(r)$. Let $\varphi = \exists y \psi(a_1, \cdots, a_n, y)$ be of the requisite form (ψ quantifier free) with $a_1, \cdots, a_n \in \mathcal{F}$ for some $\mathcal{F} \models DCF_p$, and let $b \in \mathcal{F}' \models DPF_p$, $\mathcal{F}' \supseteq \mathcal{F}$ such that $\mathcal{F}' \models \varphi(a_1, \cdots, a_n, b)$. Now let $b_1, \cdots, b_k \in \mathcal{F}'$ be all the elements corresponding to terms of the form $r(t)$ which occur in ψ. By replacing $\mathcal{F}'$ if necessary we may assume that $b_1, \cdots, b_k, b$ are each differentially algebraic over $\mathcal{F}$, since no finite bit of information such as ψ can imply that any of the terms in it correspond to differentially transcendental elements. Consider the differential field extension $\mathcal{F}\langle b_1, \cdots, b_k, b\rangle$ of $\mathcal{F}$. Since $\mathcal{F} \models DCF_p$ it must be that $\mathcal{F}$ has infinite linear dimension over its constant field (for each $n > 0$ there is a solution c_n in $\mathcal{F}$ to $D^n_y = 0$, $D^{n-1}y \neq 0$, and the set $\{c_n \mid n = 1, 2, \cdots\}$ is seen to be linearly independent over constants by checking a suitable Wronskian). Therefore $\mathcal{F}\langle b_1, \cdots, b_k, b\rangle = \mathcal{F}\langle c\rangle$ for some c, by 1.3. Now anything satisfied in $\mathcal{F}\langle c\rangle$ is satisfied in $\mathcal{F}$, by the same argument as for characteristic 0, i.e, that DCF is a 1-existentially complete theory in L. Thus DCF is a 1-existentially complete theory in L. Thus DCF is 1-existentially complete in $L(r)$ and the proof is finished.

COROLLARY 2.6. i) *In L, DCF_0 is substructure complete (i.e., admits elimination of quantifiers).*

ii) ($p \neq 0$) *In $L(r)$, DCF_p is substructure complete.*

iii) ($p = 0$ *or* $p \neq 0$, *in* $L(r)$ *or* L) *DCF_p is complete.*

PROOF. i) and ii). Immediate since each is the model completion of a universal theory in the given language.

iii) The prime algebraic field is also a prime model of DPF_p, and so DCF_p is the model completion of a theory with a prime model. It follows that DCF_p is complete.

3. Differential closure

A natural question which arises whenever one has a model-completion T^* of a theory T is whether there is any economical way to extend a model of T to one of T^*; that this is sometimes the case is seen in notion of algebraic closure and of real closure. A key property such an extension should have is that a copy of it is to be found inside any extension to a model of T^*, hence the following definition.

DEFINITION. Let T^* be the model completion of T and let $\mathscr{A} \subseteq \mathscr{B}$, $\mathscr{A} \models T$, $\mathscr{B} \models T^*$. Then $\mathscr{B}$ is a *prime model extension* of $\mathscr{A}$ provided for all $\mathscr{C} \supseteq \mathscr{A}$ such that $\mathscr{C} \models T^*$ there is an embedding of $\mathscr{B}$ into $\mathscr{C}$ over $\mathscr{A}$. Given that $\mathscr{A}$ has a prime model extension $\mathscr{B}$ there are two additional desirable properties which $\mathscr{B}$ may have, uniqueness and minimality. We say $\mathscr{B}$ is the *unique* prime model extension of $\mathscr{A}$ if any other prime model extension $\mathscr{C}$ of $\mathscr{A}$ is isomorphic to $\mathscr{B}$ over $\mathscr{A}$, and that $\mathscr{B}$ is a *minimal* prime model extension of $\mathscr{A}$ if there is no $\mathscr{C}$ with $\mathscr{A} \subseteq \mathscr{C} \subsetneq \mathscr{B}$ such that $\mathscr{C}$ is also a prime model extension.

Looking for a prime model extension involves some ability to choose which isomorphism types of extensions to include. Given a substructure complete theory T^* there is for each substructure $\mathscr{A}$ of a model of T^* an associated Stone space $S(\mathscr{A})$ of 1-type, which are just the isomorphism types of simple extensions of $\mathscr{A}$; a basic neighborhood U_φ in $S(\mathscr{A})$ is determined by a quantifier free formula $\varphi(y)$ in one free variable y; U_φ is the set of all 1-types which include $\varphi(y)$. In case U_φ is a singleton $\{q\}$ (i.e., q is an isolated point) we say that q is a principal 1-type. Since any such principal 1-type must be realized by some element in any $\mathscr{B} \supseteq \mathscr{A}$ such that $\mathscr{B} \models T^*$, it is convenient if there are a lot of principal 1-types around, as is the case for $T^* = DCF$.

THEOREM 3.1. (Blum $p = 0$, Wood and Shelah $p \neq 0$) *For $\mathscr{F} \models DPF$, the isolated points of $S(\mathscr{F})$ are dense in $S(\mathscr{F})$.*

PROOF. For characteristic 0 a 1-type is just a simple differential field extension, and the principal 1-types are exactly the isomorphism types of constrained elements, as was seen in 1.2, so the density follows from the fact that any consistent finite system of equations and inequations has a constrained solution. If the characteristic is $p \neq 0$ then we must work in $L(r)$, and the density requires a bit of information about finitely generated differential field extensions and constrained ideals in more than one variable. The fact that if $\mathscr{F} \models DPF$ and the ideal of $a_1, \cdots, a_n$ is constrained then $\mathscr{F}\langle a_1, \cdots, a_n \rangle \models DPF$ keeps the principal 1-types from being horrendous. Again we refer to [24] for details.

COROLLARY 3.2. *Over each* $\mathcal{F} \models DPF$ *there exists a prime model extension* $\tilde{\mathcal{F}} \models DCF$ *(called a differential closure of* $\mathcal{F}$*).*

PROOF. This Corollary is immediate from 3.1 by a result of Morley's, that any substructure complete theory T with the isolated points of $S(\mathcal{A})$ dense in $S(\mathcal{A})$ for all $\mathcal{A}$ a substructure of a model of T has prime model extensions for all $\mathcal{A}$. We describe this kind of prime model extension for DCF as follows: let $\mathcal{F} \models DPF$ and let $\mathcal{F}'$ be some extension of $\mathcal{F}$ to a model of DCF. Obtain a chain $\{\mathcal{F}_\alpha\}$ of models of DPF by letting $\mathcal{F}_0 = \mathcal{F}$, taking unions at limit ordinals, and taking $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha\langle b_\alpha\rangle$ for any $b_\alpha \in \mathcal{F}' - \mathcal{F}_\alpha$ which is constrained over $\mathcal{F}_\alpha$ (if such exists). The first time no such b exists we have $\mathcal{F}_\alpha = \tilde{\mathcal{F}}$, an extension of $\mathcal{F}$, and by our earlier observations it is clear that $\tilde{\mathcal{F}} \models DCF$. Further, if $\mathcal{F} \subseteq \mathcal{F}'' \models DCF$, then we can build up an isomorphism of $\tilde{\mathcal{F}}$ into $\mathcal{F}''$ using the description of $\tilde{\mathcal{F}}$ since at each stage there must be an image for b_α in any model of DCF into which $\mathcal{F}_\alpha$ is embedded (since b_α is constrained over $\mathcal{F}_\alpha$ and DCF is model complete). A prime model extension which can be written as the union of a chain of extensions of this sort (in our case extensions via constrained elements) is called a Morley prime model extension. Even without the uniqueness result we can see at this point that any differential closure $\tilde{\mathcal{F}}$ of $\mathcal{F} \models DPF$ will have the property that any $a \in \tilde{\mathcal{F}}$ realizes a principal 1-type over $\mathcal{F}$, and thus corresponds to a constrained extension (but for nonzero characteristic this may not be a simple extension). It is not however the case that for $\mathcal{F} \subseteq \mathcal{F}'$, $\mathcal{F}' \models DCF$, a differential closure of $\mathcal{F}$ is always obtainable by adjointing to $\mathcal{F}$ all elements of $\mathcal{F}'$ constrained over $\mathcal{F}$. The result will be differentially closed, but there clearly exist $\mathcal{F}'$ which contain too many ($>\operatorname{card}\mathcal{F}$) elements which realize the same constrained type of order >0.

We deal in section 4 with the uniqueness of differential closures, but first close this section with a few additional remarks about the Stone space of a differential field of characteristic 0.

DEFINITION. A theory T is *ω-stable* if for any countable $\mathcal{A} \models T$, the set $S(\mathcal{A})$ is also countable.

THEOREM 3.3. (Blum) DCF_0 *is* ω*-stable.*

PROOF. This is just a restatement of Lemma 1.1, since the elements of $S(\mathcal{F})$ correspond to simple differential field extensions of $\mathcal{F}$.

DEFINITION. The *Morley derivative* $\mathcal{D}S(\mathcal{F})$ is the set of all points in $S(\mathcal{F})$ which are either not isolated in $S(\mathcal{F})$ or which extend in $S(\mathcal{F}')$, for some $\mathcal{F}' \supseteq \mathcal{F}$, to points which are not isolated in $S(\mathcal{F}')$. The operation of taking Morley derivatives can be iterated such that for $\mathcal{F} \subseteq \mathcal{F}'$ we have $\mathcal{D}^\alpha S(\mathcal{F}') \to \mathcal{D}^\alpha S(\mathcal{F})$ under the obvious restriction map of 1-types over $\mathcal{F}'$ to 1-types over $\mathcal{F}$. (See Sacks [14] or Morley [6].) The ω-stability of a theory T implies that there exists α such that $\mathcal{D}^\alpha S(\mathcal{A}) = \emptyset$ for all $\mathcal{A}$ a substructure of a model of T; the least such α is called the *Morley rank* of T, while the *Morley rank* of $p \in S(\mathcal{A})$ is the greatest α such that $p \in \mathcal{D}^\alpha S(\mathcal{A})$, if such exists.

THEOREM 3.4. (Blum) *The Morley rank of DCF_0 is $\omega + 1$.*

PROOF. We claim first that if q is a 1-type in $S(\mathcal{F})$ corresponding to the generic zero of a polynomial f of order n, then the rank of q is at most n. Observe that for any $q' \in S(\mathcal{F}')$, where $\mathcal{F}' \supseteq \mathcal{F}$ and q' extends q, that q' corresponds to a generic zero of a polynomial of order $\leqq n$. Now for $n = 0$, q is the type of an algebraic element over $\mathcal{F}$ and is clearly isolated in $S(\mathcal{F})$ as is q' in $S(\mathcal{F}')$ for all $\mathcal{F}' \supseteq \mathcal{F}$ and $q' \supseteq q$. Now assume that $\mathcal{D}^n S(\mathcal{F})$ consists only of 1-types of generic zeros of order $\geqq n$ and let $q \in \mathcal{D}^n S(\mathcal{F})$ with corresponding f of order n. Then the formula $f(y) = 0$ together with $g(y) \neq 0$ for all g of order $< n$ determine the type q. But the formulas $g(y) \neq 0$ are superfluous in $\mathcal{D}^n S(\mathcal{F})$ since there are no types left in $\mathcal{D}^n S(\mathcal{F})$ containing $g(y) = 0$ for g of order $< n$; similarly for any $q' \supseteq q$ in $S(\mathcal{F}')$ where $\mathcal{F}' \supseteq \mathcal{F}$. Thus the formula $f(y) = 0$ isolates q in $\mathcal{D}^n S(\mathcal{F})$ and it or one of its finitely many factors over $\mathcal{F}'$ isolates a type in $\mathcal{D}^n S(\mathcal{F}')$ for $\mathcal{F}' \supseteq \mathcal{F}$, so $q \notin \mathcal{D}^{n+1} S(\mathcal{F})$. To see that some types of order n survive until the nth Morley derivative we consider the type q of a generic zero of $D^n y = 0$. For $n = 1$ we know that all constrained zeros of $Dy = 0$ are algebraic over $\mathcal{F}$, and so there can be no finite way to isolate q from the algebraic types. Assuming that the type of a generic zero of $D^{n-1} y = 0$ has rank $n - 1$, we let q_k be the 1-type of a generic zero of $D^{n-1} y = k$, for each $k > 0$, and let $a_k \in \mathcal{F}$ such that $D^{n-1} a_k = k$. Then q_k is given by saying $y - a_k$ is a generic zero of $D^{n-1} y = 0$, hence q_k also has rank $n - 1$. But now we have infinitely many elements $\{q_k \mid k > 0\}$ in $\mathcal{D}^{n-1} S(\mathcal{F})$ all containing the formula $D^n y = 0$; clearly any finite subset of q is contained in some (in fact infinitely many) q_k, so we cannot isolate q in $\mathcal{D}^{n-1} S(\mathcal{F})$ from the q_k's. Thus $q \in \mathcal{D}^n S(\mathcal{F})$. Thus we get that $\mathcal{D}^\omega S(\mathcal{F}) = \{q\}$ where q is the 1-type of the differentially transcendental extension of $\mathcal{F}$, and hence $\mathcal{D}^{\omega+1} S(\mathcal{F}) = \emptyset$, giving DCF_0 Morley rank $\omega + 1$.

Observe that this does not imply that the 1-type of a generic zero of an order n polynomial over $\mathcal{F} \models DF_0$ must survive in $\mathcal{D}^n S(\mathcal{F})$. One interesting example, due to Kovacic and Kolchin (unpublished), is that of the Painlevé transcendents; they show that given $\mathcal{F} \models DF_0$ with $x \in \mathcal{F}$ such that $Dx = 1$, the polynomial

$f(y) = D^2y - 6y^2 + x$ has no zeros of transcendence degree 1 over $\mathcal{F}$, hence that the Morley rank of the 1-type of a generic zero of f is 1, not 2.

Closer examination of 1-types over models of DF_0 lead to interesting and apparently difficult questions. First of all there is the general question of which differential polynomials have constrained generic zeros. We observe that if a is constrained over $\mathcal{F}$ then the constant field of $\mathcal{F}\langle a\rangle$ must be algebraic over $\mathcal{C}$, the constant field of $\mathcal{F}$. It is possible however to let a be differentially algebraic but not constrained over $\mathcal{F} \models DCF_0$ with $\mathcal{F}\langle a\rangle$'s constant field equal to $\mathcal{C}$; for example, if a is a generic zero of $Dy = y^3 - y^2$, by a result of Rosenlicht [11] which we state as 5.3. Also related to this is the open question of the number of countable models of DCF_0, whose solution may require additional information concerning the isomorphism types of elements which are not constrained over certain countable fields.

4. Stability and uniqueness

The uniqueness of the differential closure of a model of DF_0 follows from the ω-stability of DCF_0 by a general theorem of Shelah [21] (see also Sacks [13] or [14]). An algebraic account of this is given in Kolchin [4], where the differential closure is characterized as follows. Let $\mathcal{F} \subseteq \tilde{\mathcal{F}}$, $\mathcal{F} \models DF_0$, $\tilde{\mathcal{F}} \models DCF_0$. Then $\tilde{\mathcal{F}}$ is the differential closure of $\mathcal{F}$ if and only if every element of $\tilde{\mathcal{F}}$ is constrained over $\mathcal{F}$ and every set of indiscernibles in $\tilde{\mathcal{F}}$ over $\mathcal{F}$ is at most countable.

(A set S of elements in $\tilde{\mathcal{F}}$ is *indiscernible* over $\mathcal{F}$ if for every $n > 0$ and every formula $\varphi(x_1, \cdots, x_n)$ defined over $\mathcal{F}$ with free variables $x_1, \cdots, x_n$, either $\tilde{\mathcal{F}} \models \varphi(s_1, \cdots, s_n)$ for every n-element subset $\{s_1, \cdots, s_n\}$ of S or for none.)

For differential fields we can single out a special kind of indiscernible set S over $\mathcal{F}$, called an independent set of conjugates in Kolchin [4], with the additional property that the fields $\{\mathcal{F}\langle a\rangle \mid a \in S\}$ are linearly disjoint over $\mathcal{F}$. This implies that each $a \in S$ is a generic zero of some fixed $f \in \mathcal{F}\{y\}$ over $\mathcal{F}$ and also over any extension of the form $\mathcal{F}\langle S'\rangle$ where $S' \subseteq S - \{a\}$.

DEFINITIONS. (See [18] or [20].) A theory T is *λ-stable* if for all $\mathcal{A} \models T$ of cardinality λ, card $S(\mathcal{A}) = \lambda$. We say T is *stable* if T is λ-stable for some infinite λ; T is *superstable* if there exists κ such that T is λ-stable for all $\lambda \geqq \kappa$.

We remark that ω-stable implies λ-stable for all infinite λ, and so DCF_0 is superstable, by 3.3.

In contrast, we prove in 4.1 and 4.3 that for $p \neq 0$, DCF_p is λ-stable just in case $\lambda^{\aleph_0} = \lambda$, giving us that DCF is stable but not superstable.

THEOREM 4.1. ($p \neq 0$) *DCF_p is not superstable; for any κ and any $\mathcal{F} = DCF_p$ of cardinality κ, the cardinality of $S(\mathcal{F})$ is at least $\kappa^{\aleph_0}$.*

PROOF. Let $|\mathcal{F}| = \kappa$, $\mathcal{F} \models DCF_p$ and let $\mathcal{C}$ be the constant subfield of $\mathcal{F}$. Pick any $a \in \mathcal{F}$ such that $Da = 1$. Since $\mathcal{F}^p = \mathcal{C}$ we know $|\mathcal{C}| = \kappa$ also, so there are $\kappa^{\aleph_0}$ distinct sequences $\{c_n\}_{n \in \omega}$ of elements of $\mathcal{C}$. For each such sequence we produce a type over $\mathcal{F}$, as follows.

For b algebraically transcendental over $\mathcal{F}$ we define the differential field extension $\mathcal{F}_1 = \mathcal{F}\langle b\rangle$ by setting $Db = 1$. Then $b - a$ is a constant in $\mathcal{F}_1$ with no pth root in $\mathcal{F}_1$ (easily checked) so we may extend $\mathcal{F}_1$ to $\mathcal{F}_2 = \mathcal{F}_1\langle b_1\rangle$ where $b_1^p = b - a$, $Db_1\, Db_1 = c_1$. (In general, Db_1 can be anything we want, since $b_1 \notin \mathcal{F}_1$, $b_1^p \in \mathcal{F}_1$.) Now $b_1 - c_1 a$ is a constant of $\mathcal{F}_2$ without a pth root in $\mathcal{F}_2$ (again easy to check) and we take $\mathcal{F}_3 = \mathcal{F}_2\langle b_2\rangle$ where $b_2^p = b_1 - c_1 a$, $Db_2 = c_2$, etc. Thus the following set of formulas is consistent:

$$\{Dx = 1\} \cup \{\exists x_1 \cdots \exists x_n (x_1^p = x - a \wedge Dx_1 = c_1 \wedge x_2^p = x_1 - c_1 a \wedge Dx_2 = c_2 \wedge \cdots \wedge x_n^p = x_{n-1} - c_{n-1}\, a \wedge Dx_n = c_n) \mid n \in \omega\}$$

Each of these sets extends to obviously distinct types over $\mathcal{F}$, and there are $\kappa^{\aleph_0}$ of them.

Even though DCF_p is not superstable, there is another property, namely that of being quasi-totally transcendental, for which the Shelah proof of uniqueness of ω-stable theories goes through unchanged (e.g., for real closures, as described in Sacks [13]). A *quasi-totally transcendental* theory T is a substructure complete theory such that the Morley ranked points of $S(\mathcal{A})$ are dense in $S(\mathcal{A})$ for all substructures $\mathcal{A}$ of models of T. However, for $p \neq 0$ we have that DCF_p is not quasi-totally transcendental. For let $\mathcal{F} \models DPF_p$, $\mathcal{F}$ the prime field with p elements and let a be a zero of $Dy - 1$ over $\mathcal{F}$. Then a is constrained over $\mathcal{F}$ (any trivial constraint such as $1 \neq 0$ will do), and so $\mathcal{F}\langle a\rangle \models DPF_p$. If DCF_p were quasi-t.t., the type q of a over $\mathcal{F}$, being isolated, must also be ranked. But the set of types over $\mathcal{F}\langle a\rangle$ which contain the equation $Dy = 1$ is much too large, since we can take a generic zero b of $Dy = 1$ over $\mathcal{F}\langle a\rangle$ and we have now a new constant, $b - a$, whose pth root can have any of $2^{\aleph_0}$ types (as in the proof of 4.1), all of which goes into the 1-type of a generic zero of $Dy = 1$ over $\mathcal{F}\langle a\rangle$. Thus q cannot be ranked. The existence of these $2^{\aleph_0}$ many types implies also that there are $2^{\aleph_0}$ nonisomorphic countable differentially closed fields of characteristic $p \neq 0$. For $\kappa > \aleph_0$ the existence of 2^{κ} models of power κ follows directly from the nonsuperstability of DCF_p via a result of Shelah [18].

None of the above leads us to uniqueness of differential closure for arbitrary characteristic. For this we use a result of Shelah's stronger than that for ω-stable theories, which applies generally to stable theories for which prime model extensions exist.

LEMMA 4.2. *Let $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2 \models DCF$ such that $\mathcal{F}_1$ and $\mathcal{F}_2$ are linearly disjoint over $\mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}$, and let $\mathcal{F}' = \mathcal{F}_1\langle\mathcal{F}_2\rangle$ be a compositum of $\mathcal{F}_1$ and $\mathcal{F}_2$. Then $\mathcal{F}' \models DPF$.*

PROOF. We assume the characteristic is $p \neq 0$; otherwise there is nothing to prove. Let $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}'$ be the constant fields of $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}'$, respectively. If $c \in \mathcal{C}'$, say $c = a/b$ for polynomial expressions a, b in elements of $\mathcal{F}_1 \cup \mathcal{F}_2$, then $c = ab^{p-1}/b^p$ and $D(ab^{p-1}) = 0$, where $ab^{p-1} \in (\mathcal{F}')^p$ just in case $c \in (\mathcal{F}')^p$, so we may assume $c = \sum_{i=1}^n a_i b_i$ for $a_i \in \mathcal{F}_2$. By choosing n minimal for our given c, we conclude that the sets $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_n\}$ are each linearly independent over $\mathcal{F}$. By the disjointness of $\mathcal{F}_1$ and $\mathcal{F}_2$ over $\mathcal{F}$ it follows that the set $s = \{a_i b_j \mid 1 \leqq i, j \leqq n\}$ is also linearly independent over $\mathcal{F}$. Next we claim that for each i, Da_i is linearly dependent on $\{a_1, \cdots, a_n\}$ over $\mathcal{F}$ and similarly for the b_i's: otherwise we may assume by relabelling that $\{a_1, \cdots, a_n, Da_1, \cdots, Da_k\}$ is linearly independent over $\mathcal{F}$ and that $\{b_1, \cdots, b_n, Db_{i_1}, \cdots, Db_{i_m}\}$ is linearly independent, with all the other Da_i's and Db_j's dependent on these sets, respectively. By disjointness we know that

$$S' = S \cup \{a_i Db_{i_j} \mid 1 \leqq i \leqq n, 1 \leqq j \leqq m\} \cup \{(Da_i)b_j \mid 1 \leqq i \leqq k, 1 \leqq j \leqq n\} \cup$$

$$\{(Da_i)(Db_{i_j}) \mid 1 \leqq i \leqq k, 1 \leqq j \leqq m\}$$

is linearly independent also. But if we replace the remaining Da_i's and Db_j's respectively, by expressions in the two given linearly independent sets in the equation

$$0 = Dc = \sum_{i=1}^{n} (Da_i)b_i + \sum_{i=1}^{n} a_i(Db_i),$$

then the coefficient of $(Da_1)b_1$ is 1, contradicting the linear independence of S'. Thus we have $Da_i = \sum_{j=1}^n t_{ij} a_j$ for some $t_{i_j} \in \mathcal{F}$.

Now consider the system (*) $D_{y_i} = \sum_{j=1}^n t_{ij} y_j$, $i = 1, \cdots, n$, of linear equations over $\mathcal{F}$ in indeterminates $y_1, \cdots, y_n$. Since $\mathcal{F} \models DCF$ and (*) has a solution in $\mathcal{F}_1 \supseteq \mathcal{F}$, there exist solutions $(d_{11}, \cdots, d_{n1}), \cdots, (d_{1k}, \cdots, d_{nk})$ in $\mathcal{F}$ such that any other solution of (*) in $\mathcal{F}$ is a linear combination of the given ones. Actually, any solution $e_1, \cdots, e_n$ must actually be a linear combination over $\mathcal{C}$ of the given solutions, since if $e_i = \sum_{j=1}^k c_j d_{ij}$, $i = 1, \cdots, n$ is a solution of (*), then

$$\sum_{j=1}^{n} t_{ij}e_j = De_i = \sum_{j=1}^{k} c_j(Dd_{ij}) + \sum_{j=1}^{k} (Dc_j)d_{ij}$$

$$= \sum_{j=1}^{k} c_j\left(\sum_{j=1}^{n} t_{ij}d_{ij}\right) + \sum_{j=1}^{k} (Dc_j)d_{ij}$$

$$= \sum t_{ij}e_j + \sum_{j=1}^{k} (Dc_j)d_{ij}$$

and since the solutions were taken to be linearly independent we have $0 = \Sigma_{j=1}^{k} (Dc_j)d_{ij}$ implies $Dc_j = 0$ and so each $c_j \in \mathscr{C}$. Now since $\mathscr{F}_1 \models DCF$ this implies that each of the original a_i's can be written as a linear combination of elements of $\mathscr{F}$ with coefficients from $\mathscr{C}_1$; similarly each b_i can be written as a linear combination over $\mathscr{C}_2$ of elements of $\mathscr{F}$. Therefore $c = \Sigma_{i=1}^{m} d_i c_{i1} c_{i2}$ for some $d_i \in \mathscr{F}$, $c_{i1} \in \mathscr{C}_1$, $c_{i2} \in \mathscr{C}_2$. Again by choosing m least, we may assume $\{c_{i1}c_{i2} \mid 1 \leqq i \leqq m\}$ is linearly independent over $\mathscr{F}$. But now $0 = Dc = \Sigma_{i=1}^{m} (Dd_i)c_{i1}c_{i2}$ and so $Dd_i = 0$, giving that $c^{1/p} = \Sigma(d_i)^{1/p}(c_{i1})^{1/p}(c_{i2})^{1/p}$, hence $c \in (\mathscr{F}')^p$. This shows that $\mathscr{F}' \models DPF$; in fact we have shown that $\mathscr{C}' = \mathscr{C}_1\langle\mathscr{C}_2\rangle$.

THEOREM 4.3. (Shelah) *DCF is stable.*

PROOF. Again assume characteristic is $p \neq 0$, and let $\mathscr{F} \subseteq \mathscr{F}'$, with $\mathscr{F}, \mathscr{F}' \models DCF$ and card $\mathscr{F} \leqq \kappa$. We must check that there are $\leqq \kappa^{\aleph_0}$ 1-types realized in $\mathscr{F}'$ over $\mathscr{F}$. To each $a \in \mathscr{F}'$ we associate a pair $\mathscr{F}_{0a} \subseteq \mathscr{F}_{1a}$ of countable models of DCF such that

1) $a \in \mathscr{F}_{1a}$
2) $\mathscr{F}_{0a} = \mathscr{F}_{1a} \cap \mathscr{F}$
3) $\mathscr{F}_{1a}$ and $\mathscr{F}$ are linearly disjoint over $\mathscr{F}_{0a}$.

Clearly for any $a \in \mathscr{F}'$ such a pair exists. By Lemma 4.2, $\mathscr{F}_{1a}\langle\mathscr{F}\rangle \models DPF$. For $a, b \in \mathscr{F}'$ we consider a and b equivalent if $\mathscr{F}_{0a} = \mathscr{F}_{0b}$ and $\mathscr{F}_{1a}$ is isomorphic to $\mathscr{F}_{1b}$ over $\mathscr{F}_{0a}$ under a map which sends a to b. There are obviously $\leqq \kappa^{\aleph_0}$ such equivalence classes. If a and b are equivalent, then the isomorphism of $\mathscr{F}_{1a}$ and $\mathscr{F}_{1b}$ extends to one between $\mathscr{F}_{1a}\langle\mathscr{F}\rangle$ and $\mathscr{F}_{1b}\langle\mathscr{F}\rangle$. Since $\mathscr{F}_{1a}\langle\mathscr{F}\rangle \models DPF$ we may conclude that a and b have the same 1-type (in $L(r)$) over $\mathscr{F}$.

Macintyre has observed that the stability of each of the theories of separably closed, nonalgebraically closed, algebraic fields may be proved along the lines of this stability proof for DCF_p. This gives the first examples of stable but not superstable theories of fields.

We mention some consequences for differential closures of the general model-theoretic results concerning prime model extensions (as in Chapter 32 of Sacks [14]).

DEFINITION. Let $\mathcal{F} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$ be models of *DPF*. Then $\mathcal{F}_1$ is *normal* in $\mathcal{F}_2$ over $\mathcal{F}$ if whenever $a \in \mathcal{F}_1$, $b \in \mathcal{F}_2$ realize the same 1-type over $\mathcal{F}$, then $b \in \mathcal{F}_1$. (These include the strongly normal extensions of differential algebra.)

THEOREM 4.4. *a) Let $\tilde{\mathcal{F}}$ be a differential closure of $\mathcal{F} \models DPF$, and let $\mathcal{F}' \models DPF$, $\mathcal{F} \subseteq \mathcal{F}' \subseteq \tilde{\mathcal{F}}$. If either $\mathcal{F}'$ is finitely generated over $\mathcal{F}$ or $\mathcal{F}'$ is normal in $\tilde{\mathcal{F}}$ over $\mathcal{F}$, then $\tilde{\mathcal{F}}$ is the differential closure of $\mathcal{F}'$.*

b) Let $\mathcal{F}''$ be, constrained over $\mathcal{F} \models DPF$ (i.e., the 1-type of each element of $\mathcal{F}''$ corresponds to a constrained ideal in some finite number of differential indeterminates) and let $\mathcal{F}' \models DPF$, $\mathcal{F} \subseteq \mathcal{F}' \subseteq \mathcal{F}''$. Then either $\mathcal{F}'$ finitely generated over $\mathcal{F}$ or $\mathcal{F}'$ normal in $\mathcal{F}''$ over $\mathcal{F}$ implies $\mathcal{F}''$ is constrained over $\mathcal{F}'$.

THEOREM 4.5. (Shelah [20].) *The differential closure $\tilde{\mathcal{F}}$ of a differentially perfect differential field $\mathcal{F}$ is unique up to isomorphism over $\mathcal{F}$.*

PROOF. The proof we sketch here is for $\mathcal{F}$ of cardinality $\aleph_1$, as given in [18] for arbitrary stable theories with prime model extensions. For cardinality $\leqq \aleph_0$ uniqueness follows by the usual back and forth argument, as given by Vaught. For cardinality $\aleph_1$, the proof employs properties of closed unbounded sets and stationary sets, as given in Devlin's *Aspects of Constructability*, Springer Lecture Notes No. 354. Let $\mathcal{F} \subseteq \tilde{\mathcal{F}}$ where $\tilde{\mathcal{F}} \models DCF_p$ (p arbitrary) is the Morley prime model extension of $\mathcal{F} \models DPF_p$, with $\operatorname{card} \mathcal{F} = \operatorname{card} \tilde{\mathcal{F}} = \aleph_1$. We write $\tilde{\mathcal{F}} = \mathcal{F}\langle a_\eta \rangle_{\eta < \omega_1}$ where a_η is constrained over $\mathcal{F}\langle a_\nu \rangle_{\nu < \eta}$ with isomorphism type given by $f_\eta = 0$, $g_\eta \neq 0$. Now suppose $\mathcal{G}$ is another differential closure of $\mathcal{F}$; since $\mathcal{G}$ is embeddable in $\tilde{\mathcal{F}}$ over $\mathcal{F}$ we assume $\mathcal{F} \subseteq \mathcal{G} \subseteq \tilde{\mathcal{F}}$. Take $\{\mathcal{F}_\eta\}_{\eta < \omega_1}$ to be a continuous chain of countable fields such that

1) $\bigcup_{\eta < \omega_1} \mathcal{F}_\eta = \tilde{\mathcal{F}}$
2) $\mathcal{F}_\eta \models DCF$, $\mathcal{F}_\eta \cap \mathcal{G} \models DCF$
3) For all $\nu < \eta$, $f_\nu, g_\nu \in \mathcal{F}_\eta\{y\}$
4) For all $\nu < \eta$, $a_\nu \in \mathcal{F}_\eta$.

(This is clearly possible.)

Next we observe that there is a closed unbounded subset B of ω_1 such that for all $\eta \in B$

5) $\mathcal{F}_\eta \cap \{a_\nu \mid \nu < \omega_1\} = \{a_\nu \mid \nu < \eta\}$

and

6) For each $b \in \mathcal{F}_\eta$ the 1-type of b over $\mathcal{F}$ (necessarily principal) is given by equations and inequations with coefficients from $\mathcal{F} \cap \mathcal{F}_\eta$.

(A closed unbounded subset B of ω_1 is a cofinal subset such that $\sup(\alpha \cap B) \in B$ for all $\alpha < \omega_1$.)

Now let $B_0 = \{\eta \in B : \mathcal{G}$ is constrained over $\mathcal{F}\langle \mathcal{G} \cap \mathcal{F}_\eta \rangle\}$. We claim that B_0 contains a closed unbounded subset of ω_1.

Given the claim, i.e. given B' closed unbounded, $B' \subseteq B_0$, we build an isomorphism of $\mathcal{G}$ onto $\mathcal{F}$ by a chain of isomorphisms $\{\varphi_\eta\}_{\eta \in B'}$ such that

i) φ_η is an isomorphism from $\mathcal{F}\langle \mathcal{G} \cap \mathcal{F}_\eta \rangle$ onto $\mathcal{F}\langle \mathcal{F}_\eta \rangle$, for all $\eta \in B'$.

ii) If $\eta, \nu \in B'$, $\eta < \nu$ then $\varphi_\nu |_{\mathcal{F}\langle \mathcal{G} \cap \mathcal{F}_\eta \rangle} = \varphi_\eta$.

iii) If $\eta = \sup\{\nu \mid \nu < \eta,\ \nu \in B'\}$, then $\varphi_\eta = \bigcup\{\varphi_\nu \mid \nu < \eta,\ \nu \in B'\}$.

To define the φ_η's we need only consider the case where φ_η satisfies (i)-(iii) and ν is the B'-successor of η (or where ν is the least element of B'). Then $\mathcal{G} \cap \mathcal{F}_\nu$, $\mathcal{F}_\nu$ are countable and constrained over $(\mathcal{F} \cap \mathcal{F}_\nu)\langle \mathcal{G} \cap \mathcal{F}_\eta \rangle$, $(\mathcal{F} \cap \mathcal{F}_\nu)\langle \mathcal{F}_\eta \rangle$, respectively. The Vaught argument thus allows us to extend $\varphi_\eta |_{(\mathcal{F} \cap \mathcal{F}_\nu)\langle \mathcal{G} \cap \mathcal{F}_\eta \rangle}$ to an isomorphism from $(\mathcal{F} \cap \mathcal{F}_\nu)\langle \mathcal{G} \cap \mathcal{F}_\nu \rangle$ onto $(\mathcal{F} \cap \mathcal{F}_\nu)\langle \mathcal{F}_\nu \rangle$. This extension is consistent with the identity map on $\mathcal{F}$, and so we get that there is an isomorphism φ_ν from $\mathcal{F}\langle \mathcal{G} \cap \mathcal{F}_\eta \rangle$ onto $\mathcal{F}\langle \mathcal{F}_\eta \rangle$ which extends φ_η. The isomorphism of $\mathcal{G}$ onto $\mathcal{F}$ is then given by $\bigcup_{\eta \in B'} \varphi_\eta$.

To see the claim the procedure is to assume the contrary, i.e., that $S = \omega_1 - B_0$ is stationary. By the usual arguments about stationary sets we may conclude that there is a stationary set $S' \subseteq S$ and a pair of differential polynomials $f(y, x_1, \cdots, x_n, z_1, \cdots, z_m)$ and $g(y, x_1, \cdots, x_n, z_1, \cdots, z_m)$ — for ease of notation we assume $n = m = 1$ — defined over the prime subfield, and some $c \in \mathcal{F}_{\eta_0}$, where η_0 is the least element of S', such that: for all $\eta \in S'$ there are $a_\eta \in \mathcal{F}$ and $b_\eta \in \mathcal{G}$, b_η not principal over $\mathcal{F}\langle \mathcal{G} \cap \mathcal{F}_\eta \rangle$ but b_η constrained over $\mathcal{F}\langle \mathcal{F}_\eta \rangle$ with isomorphism type given by $f(y, a_\eta, c) = 0$, $g(y, a_\eta, c) \neq 0$. Now we choose an increasing sequence $\eta_1 < \eta_2 < \cdots$ in S' such that $a_{\eta_i}, b_{\eta_i} \in \mathcal{F}_{\eta_{i+1}}$ and pick $c_i \in \mathcal{G} \cap \mathcal{F}_{\eta_i}$ such that c_i has the same type over $\mathcal{F}\langle b_{\eta_j} \rangle_{j<i}$ as does c. (This is possible since $\mathcal{G} \cap \mathcal{F}_{\eta_i} \models DCF$ and since c realizes a principal 1-type over $\mathcal{F}$ hence over $\mathcal{F}\langle b_{\eta_j} \rangle_{j<i}$.) But now we have, for each $i < \omega$, $f(b_{\eta_k}, a_{\eta_k}, c_i) = 0$, $g(b_{\eta_k}, a_{\eta_k}, c_i) \neq 0$ if and only if $k \leqq i$. Otherwise, for $k > i$, b_{η_k} would realize a principal type over $\mathcal{F}\langle \mathcal{G} \cap \mathcal{F}_{\eta_i} \rangle$, a contradiction. Therefore the pair $f = 0$, $g \neq 0$ has what is called the order property (see [19] or [20]), which contradicts the stability of DCF_p.

5. Nonminimality of the differential closure

Recall that the differential closure $\tilde{\mathcal{F}}$ is *minimal* provided there exists no $\mathcal{F}' \models DCF$ such that $\mathcal{F} \subseteq \mathcal{F}' \subsetneqq \tilde{\mathcal{F}}$. There is a general model-theoretic criterion for minimality (see, for example, [14]) which states that $\tilde{\mathcal{F}}$ is minimal over $\mathcal{F}$ if and only if every set of indiscernibles in $\tilde{\mathcal{F}}$ over $\mathcal{F}$ is finite. Nonminimality results for characteristic 0 were obtained independently by Kolchin [4], Rosenlicht [11], and Shelah [17]; nonminimality for characteristic p is an open problem. We give here some general results due to Rosenlicht, and refer to the three papers for further details. The reader wishing to be convinced of nonminimality as easily as possible should consult §8 and the Appendix of Kolchin's paper.

A differential polynomial having an infinite set of indiscernible generic zeros must have order $\geqq 1$ and cannot be linear. Thus the natural first candidate would be something of the form $Dy - f(y)$ for, say, a nonlinear f of order 0. The following result of Rosenlicht's shows that certain of these do give rise to infinite sets of indiscernibles.

THEOREM 5.1. *Let $\mathcal{F}, \mathcal{F}' \models DF_0$, $\mathcal{F} \subseteq \mathcal{F}'$, and let $\mathcal{C}, \mathcal{C}'$ (their respective constant fields) be such that $\mathcal{C}'$ is algebraic over $\mathcal{C}$. Let $f(y) \in \mathcal{C}(y)$ (the algebraic function field over $\mathcal{C}$ in algebraic indeterminate y) where*

$$\frac{1}{f(y)} = \sum_{i=1}^{n} c_i \frac{\partial u_i/\partial y}{u_i} + \frac{\partial v}{\partial y}$$

for some $c_1, \cdots, c_n \in \mathcal{C}$, $u_1, \cdots, u_n$, $v \in \mathcal{C}(y)$.

If $a, b \in \mathcal{F}'$ are zeros of $Dy - f(y)$ and if a, b are algebraically dependent over $\mathcal{F}$, then either a or b is algebraic over $\mathcal{F}$ or $D(v(a)) = D(v(b))$.

COROLLARY 5.2. *Let $\mathcal{F}, \mathcal{F}', \mathcal{C}, \mathcal{C}'$ be as in 5.1 and let $f(y) = y/(y+1)$ or $f(y) = y^3 - y^2$. Then if a and b are distinct roots of $Dy - f(y)$, neither of which is algebraic over $\mathcal{F}$, then a and b are algebraically independent over $\mathcal{F}$.*

PROOF. For $f(y) = y/(y+1)$ take $n = 1$, $c_1 = 1$, $u_1(y) = y$, $v(y) = y$. For $f(y) = y^3 - y^2$ take $n = 1$, $c_1 = 1$, $u_1(y) = (y-1)/y$, $v(y) = 1/y$. In each case we have $D(v(a)) \neq D(v(b))$ for $a \neq b$, a, b not algebraic over $\mathcal{F}$, and so by 5.1 such a and b must be algebraically independent over $\mathcal{F}$.

We now have stated enough to yield nonminimality but we include a second interesting result of Rosenlicht's.

THEOREM 5.3. *Let $\mathcal{F} \models DF_0$, $\mathcal{C}$ its constant field, and let $\mathcal{F}' = \mathcal{F}\langle a \rangle$ where a is a generic zero of $Dy - f(y)$ for an $f(y) \in \mathcal{F}(y)$ not of the form $c(\partial u/\partial y)/u$ or $\partial v/\partial y$ for any $c \in \mathcal{C}$, $u, v \in \mathcal{C}(y)$. Then $\mathcal{C}$ is the constant field of $\mathcal{F}'$.*

Observe that 5.3 applies to the two examples given in 5.2.

THEOREM 5.4. *Let $\mathcal{C} \models DF_0$, $\mathcal{C}$ a constant field. Then the differential closure $\tilde{\mathcal{C}}$ of $\mathcal{C}$ is not minimal.*

PROOF. We note that the only constant solutions of $Dy-(y^3-y^2)=0$ are 0 and 1, hence solutions distinct from 0, 1 over $\mathscr{C}$ are not algebraic over C, and so are algebraically independent over $\mathscr{C}$. This implies that such a set of distinct zeros is indiscernible over $\mathscr{C}$, since any expression involving distinct zeros of $Dy-(y^3-y^2)$ can be reduced to an algebraic one, which is nonzero if nontrivial. Let

$$S=\{a\in\tilde{\mathscr{C}}\mid Da-(a^3-a^2)=0,\quad a\neq 0,1\}.$$

Now S is infinite, since for any $a_1,\cdots,a_n\in S$ we can find a solution of $Dy-(y^3-y^2)=0$, $y(y-1)(\prod_{i=1}^{n}(y-a_i))\neq 0$ over $\mathscr{C}$, hence in $\tilde{\mathscr{C}}\models DCF_0$. This proves nonminimality of $\tilde{\mathscr{C}}$ for general reasons, but we spell out the details in this case. Let $S'\subsetneqq S$, S' infinite, and let $\mathscr{F}=\mathscr{C}\langle S'\rangle$ be the differential field extension of $\mathscr{C}$ generated by S'. Let $\tilde{\mathscr{F}}$ be the differential closure of $\mathscr{F}$, where we assume $\mathscr{C}\subseteq\mathscr{F}\subseteq\tilde{\mathscr{F}}\subseteq\tilde{\mathscr{C}}$. By 5.3 we have that $\mathscr{C}$ is the constant field of $\mathscr{F}$. Now if $a\in S-S'$ we claim that a is not constrained over $\mathscr{F}$, hence $a\notin\tilde{\mathscr{F}}$ and so $\tilde{\mathscr{F}}\subsetneqq\tilde{\mathscr{C}}$. For we have S algebraically independent, hence a is not algebraic over $\mathscr{F}$; this says a is a generic zero of $Dy-(y^3-y^2)$ over $\mathscr{F}$, since this polynomial is irreducible and lowest among polynomials of order $\geqq 1$. Thus the isomorphism type of a over $\mathscr{F}$ is given by the set

$$\{Dy-(y^3-y^2)=0\}\cup\{y(y-1)\neq 0\}\cup\{y\neq b\mid b\in S'\}.$$

Any finite subset of this set is already realized in $\mathscr{F}$ by some element of S', and so a is not constrained over $\mathscr{F}$.

By combining 4.4 with the above result we get many differential fields whose differential closures are not minimal. Indeed, I am not aware of a nontrivial example (one not already differentially closed) of a differential field with a minimal differential closure.

The above results give also an assortment of Vaughtian pairs. We have already seen that there exist $\mathscr{F}\subsetneqq\mathscr{F}'$, $\mathscr{F},\mathscr{F}'\models DCF_0$ with $\mathscr{F}$ and $\mathscr{F}'$ having the same constant fields (as is also the case for $\tilde{\mathscr{F}}\subsetneqq\tilde{\mathscr{C}}$ in the above proof, since each has the algebraic closure of $\mathscr{C}$ as constant field). To get a pair with distinct constant fields but with some other infinite set the same, we start with constant fields $\mathscr{C}\subseteq\mathscr{C}(t)$, where t is algebraically transcendental over $\mathscr{C}$. Let S be the set of solutions of $Dy-(y^3-y^2)=0$, $y(y-1)\neq 0$ in $\tilde{\mathscr{C}}$, the differential closure of $\mathscr{C}$. If we take any bijection ψ of S onto the corresponding set of solutions in $\widetilde{\mathscr{C}(t)}$ (both are countably infinite) this extends to an isomorphism ψ of $\tilde{\mathscr{C}}$ into $\widetilde{\mathscr{C}(t)}$ over $\mathscr{C}$. This gives $\mathscr{F}=\varphi(\tilde{\mathscr{C}})\subseteq\widetilde{\mathscr{C}(t)}=\mathscr{F}'$ where the constant fields of $\mathscr{F}$ and $\mathscr{F}'$ differ: $t\notin\mathscr{F}'$ since t is not algebraic over $\mathscr{C}$. The set $\varphi(S)=$

$\{a \in \mathcal{F} \mid Da - (a^3 - a^2) = 0, a \neq 0, 1\} = \{a \in \mathcal{F}' \mid Da - (a^3 - a^2) = 0, a \neq 0, 1\}$ and so we have a second kind of Vaughtian pair.

In the setting of his nonminimality proof Shelah also concludes that for $\kappa > \aleph_0$ there are 2^κ nonisomorphic models of DCF_0 of cardinality κ. This is done by showing DCF_0 is unstable in the language $L(Q)$, where Q is the quantifier "there exists uncountably many", and then applying the central result of Shelah [19]. He shows that given $\{a_\alpha\}_{\alpha<\kappa} \cup \{b_\beta\}_{\beta<\kappa}$ differentially independent over the rationals $\mathcal{Q}$ one may adjoin a generic zero c of $Dy - a_\alpha b_\beta(y/(y+1))$ to the differential closure $\mathcal{F}$ of $\mathcal{Q}\langle\{a_\alpha\} \cup \{b_\beta\}\rangle_{\alpha,\beta<\kappa}$, and that in $\widetilde{\mathcal{F}\langle c\rangle} - \mathcal{F}$ there are no zeros of $Dy - a_\delta b_\gamma(y/(y+1))$ for $(\delta, \gamma) \neq (\alpha, \beta)$. Thus, given a binary relation R on κ, one can construct over $\mathcal{F}$ a differentially closed field $\mathcal{F}_R$ of cardinality κ such that the solution in $\mathcal{F}_R$ of $Dy - a_\alpha b_\beta(y/(y+1))$ is uncountable just in case $(\alpha, \beta) \in R$. This says DCF_0 is unstable in $L(Q)$, and the proof in [19] yields in fact that there are 2^κ nonisomorphic $\mathcal{F}_R$'s as above.

We summarize isomorphism results as follows:

	number of models of power κ	
	$\kappa = \aleph_0$	$\kappa > \aleph_0$
DCF_0	?	2^κ (as above, by Shelah [17], [19])
DCF_p, $p \neq 0$	$2^{\aleph_0}$ (see Section 4)	2^κ (since not superstable [18])

One last result we should mention is Harrington's proof [2] that the differential closure of a computable $\mathcal{F} \models DPF$ is itself computable. His proof relies on the existence of a constrained solution to any finite system of equations and inequations. In connection with this and also with the discussion of differential fields in Robinson's 1973 address [9] we note that it is not known whether one can actually compute a constrained polynomial with a given constraint and satisfying a given finite system of equations. This is one aspect of the long outstanding Ritt problem; it is not even clear that this is a question of logic at all, if it were to have a positive solution. An answer would nonetheless aid in the actual description of the differential closure of a given field, something which might be viewed as desirable.

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References

1. L. Blum, *Generalized algebraic structures: A model theoretic approach*, Ph.D. Thesis, Massachusetts Institute of Technology, 1968.

2. L. Harrington, *Recursively presentable prime models*, J. Symbolic Logic **39** (1974), 305–309.

3. I. Kaplansky, *An Introduction to Differential Algebra*, Hermann, Paris, 1957.

4. E. R. Kolchin, *Constrained extensions of differential fields*, Advances in Math. **12** (1974), 141–170.

5. E. R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, New York, 1973.

6. M. Morley, *Categoricity in power*, Trans. Amer. Math. Soc. **114** (1965), 514–538.

7. J. F. Ritt, *Differential Algebra*, Amer. Math. Soc. Colloq. Publ., Vol. 33, New York, 1950.

8. A. Robinson, *An Introduction to Model Theory*, North-Holland, Amsterdam, 1965.

9. A. Robinson, *On the concept of a differentially closed field*, Bull. Res. Counc. Isr. **8F** (1959), 113–128.

10. A. Robinson, *Metamathematical problems*, J. Symbolic Logic **38** (1973), 500–516.

11. M. Rosenlicht, *The nonminimality of the differential closure*, Pacific J. Math. **52** (1974), 529–537.

12. G. Sacks, *The differential closure of a differential field*, Bull. Amer. Math. Soc. **78** (1972), 629–634.

13. G. Sacks, *Model theory and applications*, C.I.M.E. Lecture Notes, July 1975.

14. G. Sacks, *Saturated Model Theory*, Benjamin, Reading, Mass., 1972.

15. A. Seidenberg, *An elimination theory for differential algebra*, Univ. Calif. Math. Publications **3** (1956), 31–65.

16. A. Seidenberg, *Some basic theorems in differential algebra*, Trans. Amer. Math. Soc. **73** (1952), 174–190.

17. S. Shelah, *Differentially closed fields*, Israel J. Math. **16** (1973), 314–328.

18. S. Shelah, *The lazy model-theoretician's guide to stability*, Univ. of Louvain, 1975.

19. S. Shelah, *The number of non-isomorphic models of an unstable first-order theory*, Israel J. Math. **9** (1971), 473–487.

20. S. Shelah, *Stability and the number of nonisomorphic models*, North-Holland, to appear.

21. S. Shelah, *Uniqueness and characterization of prime models over sets for totally transcendental first-order theories*, J. Symbolic Logic **37** (1972), 107–113.

22. A. Weil, *Foundations of Algebraic Geometry*, Amer. Math. Soc., 1962.

23. C. Wood, *The model theory of differential fields of characteristic $p \neq 0$*, Proc. Amer. Math. Soc. **40** (1973), 577–584.

24. C. Wood, *Prime model extensions for differential fields of characteristic $p \neq 0$*, J. Symbolic Logic **39** (1974), 469–477.

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